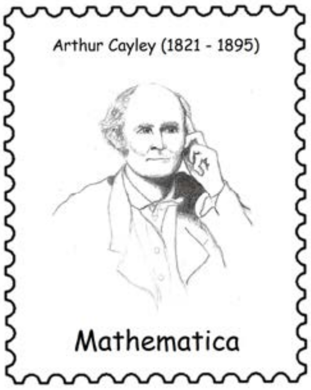
ARTHUR CAYLEY (August 16, 1821 – January 26, 1895)

by HEINZ KLAUS STRICK, Germany



Born in Richmond (Surrey), ARTHUR CAYLEY, the son of the English merchant HENRY CAYLEY, initially grew up in St Petersburg, Russia. When the family returned to England, he attended King's College in London.

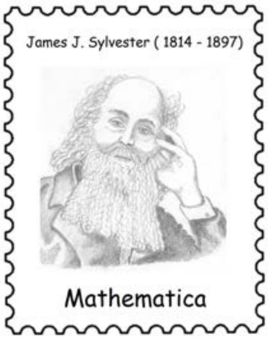
He entered Trinity College, Cambridge University at 17, graduated in mathematics at the age of 21 (as *senior wrangler*) and won the *Smith's Prize*, an award for students who have shown exceptional performance throughout their studies.

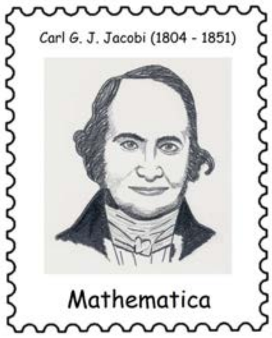
During his studies, he published three articles in the *Cambridge Mathematical Journal*. In the two years after the bachelor's degree, during which he supervised new students as a tutor, there were another 28 contributions. After passing his master's degree, he decided to become a lawyer in order to earn a better living. In the following five years he worked for a well-known London notary before he was admitted to the bar in 1849.

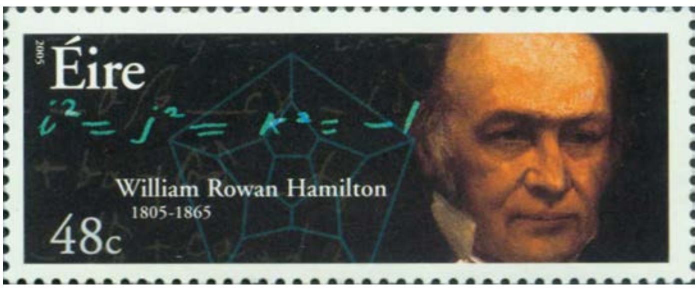
During this time he met JAMES JOSEPH SYLVESTER, who had the same interests as him. The two became friends; their conversations were always and almost exclusively about mathematical topics.

CAYLEY worked as a lawyer for 14 years – and published over 250 scientific papers during this time, before he was appointed to the SADLEIRian Chair, a chair for pure mathematics at the University of Cambridge, in 1863. Although he now had only a fraction of the income he previously had as a lawyer, he was happy with the work he was able to do until his death in 1895.

In total, he published 967 articles that dealt with topics from all current research areas of mathematics, and including one textbook (on JACOBI's elliptical functions). In contrast to many other English scientists, he also published numerous articles in French, which he mastered effortlessly.







(drawings: © Andreas Strick)

In 1843, the Irish mathematician and physicist WILLIAM ROWAN HAMILTON had dealt with 4-dimensional objects (a; b; c; d) and found with this system multiplication could only be defined if one replaced the property of commutativity with anti-commutativity.

The following rules for multiplying base units (i, j, k) apply: $i^2 = j^2 = k^2 = i \cdot j \cdot k = -1$, $i \cdot j = -j \cdot i = k$. These quaternions can also be written in the form $a + b \cdot i + c \cdot j + d \cdot k$.

In the same year, HAMILTON's friend JOHN THOMAS GRAVES found out that 8-dimensional objects (which he called *octaves*) could also be multiplied if one renounced not only the property of commutativity, but also that of associativity .

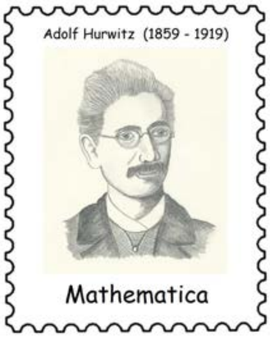
For the associated base units (*i*, *j*, *k*, *l*, *m*, *n*, *o*) the following multiplication rules apply:

$i^2 = j^2 = k^2 = l^2 = m^2 = n^2 = o^2 = i \cdot j \cdot k = -1$, $m = i \cdot l = -l \cdot i$, $n = j \cdot l = -l \cdot j$ and $o = k \cdot l = -l \cdot k$.

Since HAMILTON failed to publicise his friend's discovery, it happened two years later that CAYLEY rediscovered these hyper-complex numbers independently of GRAVES and wrote a treatise on them. Despite GRAVES' undisputed priority, the numbers are now known as *CAYLEY numbers* and are designated with CAYLEY's term *octonions*.

GRAVES and CAYLEY searched in vain for any additional similar *n*-dimensional number systems. GRAVES did not succeed for the case $n = 16$ and CAYLEY also failed for other powers of two.

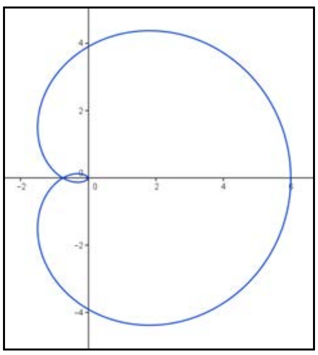
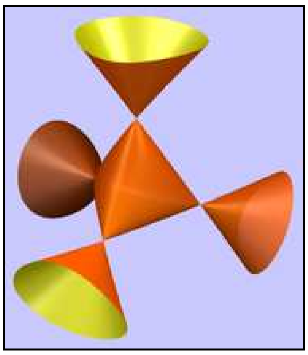
ADOLF HURWITZ was the first to prove in 1898 that, apart from $n = 1, 2, 4$ and 8, there were no other number systems with comparable multiplication rules.



In 1773, LAGRANGE asked what properties of a binary quadratic form of the type $ax^2 + bxy + cy^2$ were not changed by a transformation, i.e. were *invariant*.

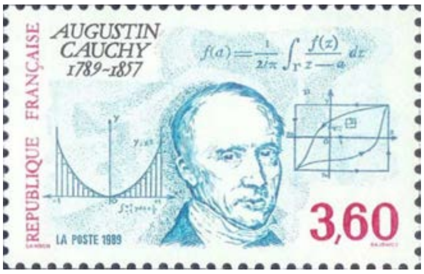
In the meantime, differential and integral calculus methods had been added to the originally purely algebraic methods. In addition geometric questions also played an important role, which CAYLEY, in particular, was confidently developing. From 1845 to 1878 he published a series of ten papers on invariant theory (*Memoirs on Quantics*). His friend SYLVESTER complemented these with his own contributions. It is not without reason that the two were often referred to as the *invariant twins* by contemporaries.

Among other things, CAYLEY's invariance studies led to a better understanding of the relationship between classic EUCLIDean geometry and the various models of non-EUCLIDean geometry. CAYLEY had dealt with the characterization of curves and surfaces in numerous publications.



(picture on the right: Wikipedia / Salix alba)

The figure on the left shows CAYLEY's sextic (equation: $4 \cdot (x^2 + y^2 - ax)^3 = 27a^2 \cdot (x^2 + y^2)^2$.
The figure on the right shows a 3-dimensional projection of CAYLEY's surface (equation: $wxy + xyz + yzw + zwx = 0$).

Another area that CAYLEY was developing was that of *group theory*, based on the work of AUGUSTIN CAUCHY and ÉVARISTE GALOIS. In 1815, CAUCHY introduced the parenthesis notation that is still used today for permutations of mathematical objects (for example, numbers), in which the initial arrangement is at the top, the modified arrangement at the bottom. He labeled the individual permutations with letters; the successive composition could then be considered as a product of these identifiers. In a document from 1844, CAUCHY continued this theory and coined the concept of *identity* and *inverse* permutation.



For example, if you consider the possible arrangements of the numbers 1, 2, 3, the following six permutations can be defined:

$$p_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} ;\ p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} ;\ p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} ;\ p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} ;\ p_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} ;\ p_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

The element p_0 is the identity: the neutral element for the sequential execution of permutations. Multiplying two permutations is not commutative; for example:

$p_1 \circ p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = p_3$, but on the other hand:

$$p_2 \circ p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = p_4$$

For each element of $S_3 = \{p_0, p_1, p_2, p_3, p_4, p_5\}$ there is an inverse element under composition. For example $p_3 \circ p_4 = p_0$ and $p_4 \circ p_3 = p_0$, i.e. $p_3 = p_4^{-1}$ and $p_4 = p_3^{-1}$.

The following subsets of S_3 are closed with respect to $\circ$ and form subgroups of S_3:
$\{p_0\}$, $\{p_0, p_1\}$, $\{p_0, p_2\}$, $\{p_0, p_5\}$, $\{p_0, p_3, p_4\}$.

CAUCHY had recognised that the *order* (= number of elements) of a subgroup must be a divisor of the order of the group. In the example of S_3 with 3! = 6 elements there are subgroups with 1, 2, 3 elements (see above) and, since each group is a subgroup of itself, there is also a subgroup with six elements, namely S_3 itself.

In a work from 1854, CAYLEY showed that for every finite group G with n elements there is a subgroup of the group S_n (= group of all permutations of n objects, i.e. with n! elements), which has the same structure – CAYLEY described this property as *faithful*, or as we would now say: *isomorphic*.

CAYLEY was the first to get an overview of the elements and their possible links in tabular form (CAYLEY group table), as an example on the right the group table for S_3.

$\circ$	p_0	p_1	p_2	p_3	p_4	p_5
p_0	p_0	p_1	p_2	p_3	p_4	p_5
p_1	p_1	p_0	p_3	p_2	p_5	p_4
p_2	p_2	p_4	p_0	p_5	p_1	p_3
p_3	p_3	p_5	p_1	p_4	p_0	p_2
p_4	p_4	p_2	p_5	p_0	p_3	p_1
p_5	p_5	p_3	p_4	p_1	p_2	p_0

Multiplication tables gave him the opportunity to find out which types of groups exist at all through systematic considerations. So he discovered that there are two possible types of groups with four elements {*e*, *a*, *b*, *c*}.

$\circ$	*e*	*a*	*b*	*c*
e	*e*	*a*	*b*	*c*
a	*a*	*e*	*c*	*b*
b	*b*	*c*	*e*	*a*
c	*c*	*b*	*a*	*e*

$\circ$	*e*	*a*	*b*	*c*
e	*e*	*a*	*b*	*c*
a	*a*	*b*	*c*	*e*
b	*b*	*c*	*e*	*a*
c	*c*	*e*	*a*	*b*

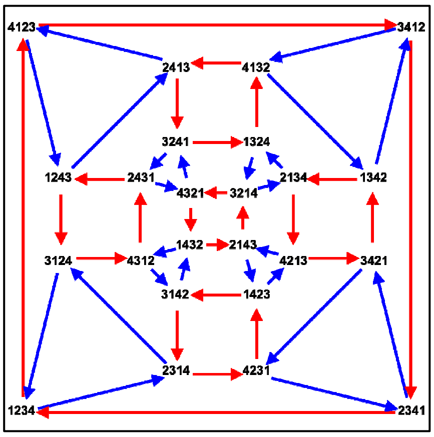
On the left you can note that the element *c* is also $a \circ b$ (this is the KLEIN *group* of order four).

On the right there is: $a \circ a = a^2 = b$ and $a \circ b = a \circ a^2 = a^3 = c$ (this is the *cyclic group* generated by the element *a*).

In a later paper he developed the idea of using *graphs* to illustrate the group structures.

The figure on the right shows the structure of the group S_4 of the permutations of four objects, the colored arrows corresponding to different types of permutations.

(source: Wikipedia watchduck)



CAYLEY also made special contributions to the development of the theory of matrices. In 1858 he published a treatise in which he examined the matrix calculus with regard to algebraic laws. With regard to the product of matrices as a multiplication, he introduced the concept of the inverse matrix.

Together with HAMILTON, he proved that every square matrix A satisfied the characteristic equation $\det(A - \lambda \cdot E) = 0$, where *E* is the unit matrix and *det* the determinant function.

So we have

$$\det(A - \lambda \cdot E) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)\cdot(d-\lambda) - b\cdot c = \lambda^2 - (a+d)\cdot\lambda - b\cdot c$$

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $A^2 = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}$ this means:

If you use A for λ, you get the zero matrix (this is called the CAYLEY-HAMILTON theorem):

$$A^2 - (a+d)\cdot A - b\cdot c\cdot E = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

CAYLEY also used the matrix calculus in connection with the quaternions. It described the basic units I, J, K as complex 2×2 matrices:

$$I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix};\; J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};\; K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix};\; E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

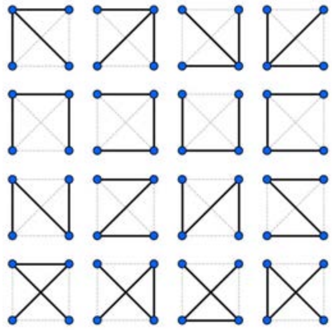
He confirmed the validity of the multiplication given above for these matrices. Using this matrix representation, he recognized that the rotation of objects in space could be described using quaternions.

One of the propositions that can be found in CAYLEY's work is called CAYLEY's formula, a theorem in combinatorics:

Consider n nodes that are connected by n-1 edges.

Then there are n^{n-2} different spanning trees on these n nodes.

(source: Wikipedia / Quartl)



In addition to topics from various areas of pure mathematics, CAYLEY made significant contributions to theoretical mechanics and perturbation theory in astronomy. He also solved a question that arose in connection with solar eclipses:

- Which curve describes the area of the shadow of the moon on the curved surface of the earth?

In 1889, the *Cambridge University Press* asked CAYLEY to compile his 967 papers (more than EULER in number) with a view to publication as a whole. In the last six years of his life he completed seven of a total of 13 volumes; his successor at the chair in Cambridge completed the work.

After CAYLEY's death, scientists from many countries gathered for the funeral service in Trinity College Chapel including even mathematicians from Russia and the United States.

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https://www.spektrum.de/wissen/arthur-cayley-der-anwalt-mit-dem-unstillbaren-hang-zur-mathematik/1324258

Translated 2020 by John O’Connor, University of St Andrews

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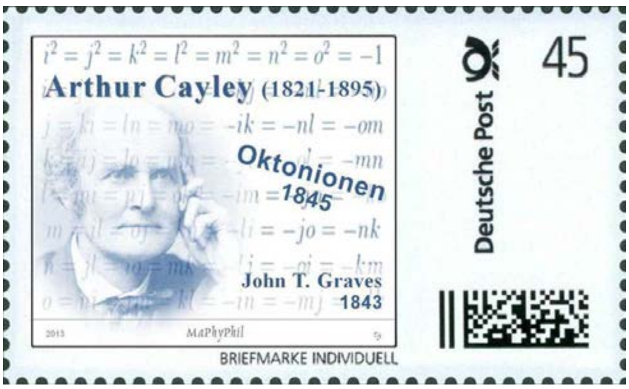

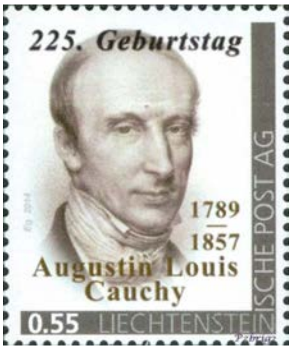

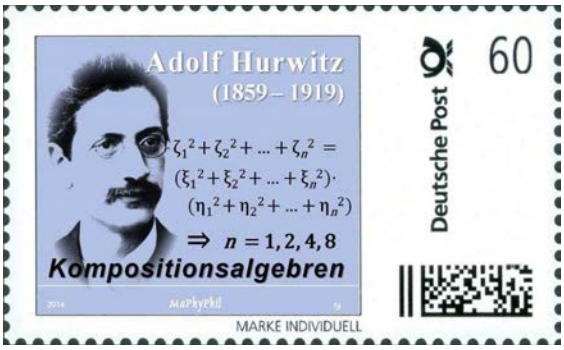

*	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1